

Constructing New Braided T -Categories via Weak Monoidal Hom-Hopf Algebras

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ABSTRACT

In this paper, we define and study weak monoidal Hom-Hopf algebras, which generalize both weak Hopf algebras and monoidal Hom-Hopf algebras. If H is a weak monoidal Hom-Hopf algebra with bijective antipode and let $Aut_{wmHH}(H)$ be the set of all automorphisms of H . Then we introduce a category ${}_H\mathcal{WMHYD}^H(\alpha, \beta)$ with $\alpha, \beta \in Aut_{wmHH}(H)$ and construct a braided T -category $\mathcal{WMHYD}(H)$ that having all the categories ${}_H\mathcal{WMHYD}^H(\alpha, \beta)$ as components.

Key words: weak monoidal Hom-Hopf algebra; braided T -category; weak (α, β) -Yetter-Drinfeld category.

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Introduction

Hom-algebras first appeared in [4], where the associativity was replaced by the Hom-associativity and similar to the Hom-coassociativity (see in [3, 5]). Based on these properties, definitions of Hom-bialgebras, Hom-Hopf algebras and further developments existed later in [1], [2]-[5], [6], [7], [13] and [16]. In [13], the authors illustrated Hom-structures from the point of view of monoidal categories and introduced monoidal Hom-algebras, monoidal Hom-coalgebras, etc., in a symmetric monoidal category, which were different from the Hom-algebras and Hom-coalgebras.

Weak Hom-Hopf algebra was introduced in [19]. The axioms were the same as the ones for a Hom-Hopf algebra, except that the coproduct of the unit, the product of counit and the antipode condition were replaced by the weaker properties.

Turaev in [15, 14] generalized quantum invariants of 3-manifolds to the case of a 3-manifold M endowed with a homotopy class of maps $M \rightarrow K(G, 1)$, where G is a group. And braided

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T -categories which are braided monoidal categories in Freyd-Yetter categories of crossed G -sets (see in [11]) play a key role of constructing these homotopy invariants.

Then one main question naturally arises. How to construct classes of new braided T -categories?

The main propose of this paper is to generalize the notion of monoidal Hom-Hopf algebras to weak cases. We first define the notion of weak monoidal Hom-bigebras and weak monoidal Hom-Hopf algebras. And we give examples to show its not a trivial generalization of monoidal Hom-Hopf algebras. We also investigate the relationship between weak monoidal Hom-Hopf algebras and weak Hom-Hopf algebras. After that, we construct new examples of braided T -categories, which generalize the construction given by Panaite and Staic (see in [12]).

The article is organized as follows.

In Section 1, we recall definitions and basic results related to monoidal Hom-Hopf algebras and braided T -categories. In Section 2, we apply the construction given by Stef and Isar [13] to the category of vector spaces, then we give definitions of weak monoidal Hom-bialgebras and some axioms which will be used in Section 3 and Section 4. We also define the weak monoidal Hom-Hopf algebras and give some properties of its antipode.

In Section 3, we introduce a class of new categories ${}_H\mathcal{WMHYD}^H(\alpha, \beta)$ (see Definition 3.1) of weak (α, β) -Yetter-Drinfeld modules associated with $\alpha, \beta \in \text{Aut}_{wmHH}(H)$. Furthermore, we prove that the category ${}_H\mathcal{WMHYD}^H(\alpha, \beta)$ is actually a weak monoidal entwined Hom-module category ${}_H\mathcal{M}^H(\psi(\alpha, \beta))$. Then in Section 4, we prove ${}_H\mathcal{WMHYD}^H$ is a monoidal category and then construct a class of new braided T -categories ${}_H\mathcal{WMHYD}^H$ in the sense of Turaev[14].

1. Preliminaries

Throughout, let k be a fixed field. Everything is over k unless otherwise specified. We refer the readers to the books of Sweedler [9] for the relevant concepts on the general theory of Hopf algebras. Let (C, Δ) be a coalgebra. We use the Sweedler-Heyneman's notation for Δ as follows:

$$\Delta(c) = \sum c_1 \otimes c_2,$$

for all $c \in C$.

1.1. Monoidal Hom-Hopf algebras.

Let $\mathcal{M}_k = (\mathcal{M}_k, \otimes, k, a, l, r)$ denote the usual monoidal category of k -vector spaces and linear maps between them. Recall from [13] that there is the *monoidal Hom-category*

$\tilde{\mathcal{H}}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (k, id), \tilde{a}, \tilde{l}, \tilde{r})$, a new monoidal category, associated with \mathcal{M}_k as follows:

- The objects of $\mathcal{H}(\mathcal{M}_k)$ are couples (M, μ) , where $M \in \mathcal{M}_k$ and $\mu \in Aut_k(M)$, the set of all k -linear automorphisms of M ;
- The morphism $f : (M, \mu) \rightarrow (N, \nu)$ in $\mathcal{H}(\mathcal{M}_k)$ is the k -linear map $f : M \rightarrow N$ in \mathcal{M}_k satisfying $\nu \circ f = f \circ \mu$, for any two objects $(M, \mu), (N, \nu) \in \mathcal{H}(\mathcal{M}_k)$;
- The tensor product is given by

$$(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)$$

for any $(M, \mu), (N, \nu) \in \mathcal{H}(\mathcal{M}_k)$.

- The tensor unit is given by (k, id) ;
- The associativity constraint \tilde{a} is given by the formula

$$\tilde{a}_{M,N,L} = a_{M,N,L} \circ ((\mu \otimes id) \otimes \varsigma^{-1}) = (\mu \otimes (id \otimes \varsigma^{-1})) \circ a_{M,N,L},$$

for any objects $(M, \mu), (N, \nu), (L, \varsigma) \in \mathcal{H}(\mathcal{M}_k)$;

- The left and right unit constraint \tilde{l} and \tilde{r} are given by

$$\tilde{l}_M = \mu \circ l_M = l_M \circ (id \otimes \mu), \quad \tilde{r}_M = \mu \circ r_M = r_M \circ (\mu \otimes id)$$

for all $(M, \mu) \in \mathcal{H}(\mathcal{M}_k)$.

We now recall from [13] the following notions used later.

A *unital monoidal Hom-associative algebra* (a monoidal Hom-algebra in Proposition 2.1 of [13]) is a vector space A together with an element $1_A \in A$ and linear maps

$$m : A \otimes A \rightarrow A; \quad a \otimes b \mapsto ab, \quad \alpha \in Aut_k(A)$$

such that

$$\alpha(a)(bc) = (ab)\alpha(c), \tag{1.1}$$

$$\alpha(ab) = \alpha(a)\alpha(b),$$

$$a1_A = 1_A a = \alpha(a), \tag{1.2}$$

$$\alpha(1_A) = 1_A, \tag{1.3}$$

for all $a, b, c \in A$.

Remark. (1) In the language of Hopf algebras, m is called the Hom-multiplication, α is the twisting automorphism and 1_A is the unit. Note that Eq.(1.1) can be rewritten as $a(b\alpha^{-1}(c)) = (\alpha^{-1}(a)b)c$. The monoidal Hom-algebra A with α will be denoted by (A, α) .

(2) Let (A, α) and (A', α') be two monoidal Hom-algebras. A monoidal Hom-algebra map $f : (A, \alpha) \rightarrow (A', \alpha')$ is a linear map such that $f \circ \alpha = \alpha' \circ f, f(ab) = f(a)f(b)$ and $f(1_A) = 1_{A'}$.

(3) The definition of monoidal Hom-algebras is different from the unital Hom-associative algebras in [3] and [5] in the following sense. The same twisted associativity condition (1.1) holds in both cases. However, the unitality condition in their notion is the usual untwisted

one: $a1_A = 1_A a = a$, for any $a \in A$, and the twisting map α does not need to be monoidal (that is, (1.2) and (1.3) are not required).

A counital monoidal Hom-coassociative coalgebra is an object (C, γ) in the category $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with linear maps $\Delta : C \rightarrow C \otimes C$, $\Delta(c) = c_1 \otimes c_2$ and $\varepsilon : C \rightarrow k$ such that

$$\gamma^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \gamma^{-1}(c_2), \quad (1.4)$$

$$\Delta(\gamma(c)) = \gamma(c_1) \otimes \gamma(c_2), \quad (1.5)$$

$$\begin{aligned} c_1 \varepsilon(c_2) &= \gamma^{-1}(c) = \varepsilon(c_1) c_2, \\ \varepsilon(\gamma(c)) &= \varepsilon(c) \end{aligned} \quad (1.6)$$

for all $c \in C$.

Remark. (1) Note that (1.4) is equivalent to $c_1 \otimes c_{21} \otimes \gamma(c_{22}) = \gamma(c_{11}) \otimes c_{12} \otimes c_2$. Analogue to monoidal Hom-algebras, monoidal Hom-coalgebras will be short for counital monoidal Hom-coassociative coalgebras without any confusion.

(2) Let (C, γ) and (C', γ') be two monoidal Hom-coalgebras. A monoidal Hom-coalgebra map $f : (C, \gamma) \rightarrow (C', \gamma')$ is a linear map such that $f \circ \gamma = \gamma' \circ f$, $\Delta \circ f = (f \otimes f) \circ \Delta$ and $\varepsilon' \circ f = \varepsilon$.

A monoidal Hom-bialgebra $H = (H, \alpha, m, 1_H, \Delta, \varepsilon)$ is a bialgebra in the monoidal category $\tilde{\mathcal{H}}(\mathcal{M}_k)$. This means that $(H, \alpha, m, 1_H)$ is a monoidal Hom-algebra and $(H, \alpha, \Delta, \varepsilon)$ is a monoidal Hom-coalgebra such that Δ and ε are morphisms of algebras, that is, for all $h, g \in H$,

$$\Delta(hg) = \Delta(h)\Delta(g), \quad \Delta(1_H) = 1_H \otimes 1_H, \quad \varepsilon(hg) = \varepsilon(h)\varepsilon(g), \quad \varepsilon(1_H) = 1.$$

A monoidal Hom-bialgebra (H, α) is called a *monoidal Hom-Hopf algebra* if there exists a morphism (called antipode) $S : H \rightarrow H$ in $\tilde{\mathcal{H}}(\mathcal{M}_k)$ (i.e., $S \circ \alpha = \alpha \circ S$), which is the convolution inverse of the identity morphism id_H (i.e., $S * id = 1_H \circ \varepsilon = id * S$). Explicitly, for all $h \in H$,

$$S(h_1)h_2 = \varepsilon(h)1_H = h_1S(h_2).$$

Remark. (1) Note that a monoidal Hom-Hopf algebra is by definition a Hopf algebra in $\tilde{\mathcal{H}}(\mathcal{M}_k)$.

(2) Furthermore, the antipode of monoidal Hom-Hopf algebras has almost all the properties of antipode of Hopf algebras such as

$$S(hg) = S(g)S(h), \quad S(1_H) = 1_H, \quad \Delta(S(h)) = S(h_2) \otimes S(h_1), \quad \varepsilon \circ S = \varepsilon.$$

That is, S is a monoidal Hom-anti-(co)algebra homomorphism. Since α is bijective and commutes with S , we can also have that the inverse α^{-1} commutes with S , that is, $S \circ \alpha^{-1} = \alpha^{-1} \circ S$.

In the following, we recall the notions of actions on monoidal Hom-algebras and coactions on monoidal Hom-coalgebras.

Let (A, α) be a monoidal Hom-algebra. A *left (A, α) -Hom-module* consists of an object (M, μ) in $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with a morphism $\psi : A \otimes M \rightarrow M, \psi(a \otimes m) = a \cdot m$ such that

$$\alpha(a) \cdot (b \cdot m) = (ab) \cdot \mu(m), \quad \mu(a \cdot m) = \alpha(a) \cdot \mu(m), \quad 1_A \cdot m = \mu(m),$$

for all $a, b \in A$ and $m \in M$.

Monoidal Hom-algebra (A, α) can be considered as a Hom-module on itself by the Hom-multiplication. Let (M, μ) and (N, ν) be two left (A, α) -Hom-modules. A morphism $f : M \rightarrow N$ is called left (A, α) -linear if $f(a \cdot m) = a \cdot f(m), f \circ \mu = \nu \circ f$. We denoted the category of left (A, α) -Hom modules by $\tilde{\mathcal{H}}_A(\mathcal{M}_k)$.

Similarly, let (C, γ) be a monoidal Hom-coalgebra. A *right (C, γ) -Hom-comodule* is an object (M, μ) in $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k -linear map $\rho_M : M \rightarrow M \otimes C, \rho_M(m) = m_{(0)} \otimes m_{(1)}$ such that

$$\mu^{-1}(m_{(0)}) \otimes \Delta_C(m_{(1)}) = (m_{(0)(0)} \otimes m_{(0)(1)}) \otimes \gamma^{-1}(m_{(1)}), \quad (1.7)$$

$$\rho_M(\mu(m)) = \mu(m_{(0)}) \otimes \gamma(m_{(1)}), \quad m_{(0)} \varepsilon(m_{(1)}) = \mu^{-1}(m), \quad (1.8)$$

for all $m \in M$.

(C, γ) is a Hom-comodule on itself via the Hom-comultiplication. Let (M, μ) and (N, ν) be two right (C, γ) -Hom-comodules. A morphism $g : M \rightarrow N$ is called right (C, γ) -colinear if $g \circ \mu = \nu \circ g$ and $g(m_{(0)}) \otimes m_{(1)} = g(m)_{(0)} \otimes g(m)_{(1)}$. The category of right (C, γ) -Hom-comodules is denoted by $\tilde{\mathcal{H}}(\mathcal{M}^C)$.

Let (H, α) be a monoidal Hom-bialgebra. We now recall from [17] that a monoidal Hom-algebra (B, β) is called a *left H -Hom-module algebra*, if (B, β) is a left H -Hom-module with action \cdot obeying the following axioms:

$$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b), \quad h \cdot 1_B = \varepsilon(h)1_B, \quad (1.9)$$

for all $a, b \in B, h \in H$.

Recall from [8] that a monoidal Hom-algebra (B, β) is called a *left H -Hom-comodule algebra*, if (B, β) is a left H -Hom-comodule with coaction ρ obeying the following axioms:

$$\rho(ab) = a_{(-1)}b_{(-1)} \otimes a_{(0)}b_{(0)}, \quad \rho(1_B) = 1_B \otimes 1_H,$$

for all $a, b \in B, h \in H$.

Let (H, m, Δ, α) be a monoidal Hom-bialgebra. Recall from ([8, 18]) that a *left-right Yetter-Drinfeld Hom-module* over (H, α) is the object (M, \cdot, ρ, μ) which is both in $\tilde{\mathcal{H}}_H(\mathcal{M})$ and $\tilde{\mathcal{H}}(\mathcal{M}^H)$ obeying the compatibility condition:

$$h_1 \cdot m_{(0)} \otimes h_2 m_{(1)} = (\alpha(h_2) \cdot m)_{(0)} \otimes \alpha^{-1}(\alpha(h_2) \cdot m)_{(1)} h_1. \quad (1.10)$$

Remark. (1) The category of all left-right Yetter-Drinfeld Hom-modules is denoted by $\tilde{\mathcal{H}}_{(H)\mathcal{YD}^{\mathcal{H}}}$ with understanding morphism.

(2) If (H, α) is a monoidal Hom-Hopf algebra with a bijective antipode S , then the above equality is equivalent to

$$\rho(h \cdot m) = \alpha(h_{21}) \cdot m_{(0)} \otimes (h_{22}\alpha^{-1}(m_{(1)}))S^{-1}(h_1),$$

for all $h \in H$ and $m \in M$.

1.2. Braided T -categories.

A *monoidal category* $\mathcal{C} = (\mathcal{C}, \mathbb{I}, \otimes, a, l, r)$ is a category \mathcal{C} endowed with a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ (the *tensor product*), an object $\mathbb{I} \in \mathcal{C}$ (the *tensor unit*), and natural isomorphisms a (the *associativity constraint*), where $a_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ for all $U, V, W \in \mathcal{C}$, and l (the *left unit constraint*) where $l_U : \mathbb{I} \otimes U \rightarrow U$, r (the *right unit constraint*) where $r_U : U \otimes \mathbb{I} \rightarrow U$ for all $U \in \mathcal{C}$, such that for all $U, V, W, X \in \mathcal{C}$, the *associativity pentagon* $a_{U,V,W \otimes X} \circ a_{U \otimes V, W, X} = (U \otimes a_{V,W,X}) \circ a_{U,V \otimes W, X} \circ (a_{U,V,W} \otimes X)$ and $(U \otimes l_V) \circ (r_U \otimes V) = a_{U,\mathbb{I},V}$ are satisfied. A monoidal category \mathcal{C} is *strict* when all the constraints are identities.

Let G be a group and let $Aut(\mathcal{C})$ be the group of invertible strict tensor functors from \mathcal{C} to itself. A category \mathcal{C} over G is called a *crossed category* if it satisfies the following:

- \mathcal{C} is a monoidal category;
- \mathcal{C} is disjoint union of a family of subcategories $\{\mathcal{C}_\alpha\}_{\alpha \in G}$, and for any $U \in \mathcal{C}_\alpha$, $V \in \mathcal{C}_\beta$, $U \otimes V \in \mathcal{C}_{\alpha\beta}$. The subcategory \mathcal{C}_α is called the α th component of \mathcal{C} ;
- Consider a group homomorphism $\varphi : G \rightarrow Aut(\mathcal{C})$, $\beta \mapsto \varphi_\beta$, and assume that $\varphi_\beta(\varphi_\alpha) = \varphi_{\beta\alpha\beta^{-1}}$, for all $\alpha, \beta \in G$. The functors φ_β are called conjugation isomorphisms.

Furthermore, \mathcal{C} is called strict when it is strict as a monoidal category.

Left index notation: Given $\alpha \in G$ and an object $V \in \mathcal{C}_\alpha$, the functor φ_α will be denoted by ${}^V(\cdot)$, as in Turaev [14] or Zunino [10], or even ${}^\alpha(\cdot)$. We use the notation $\overline{V}(\cdot)$ for ${}^{\alpha^{-1}}(\cdot)$. Then we have ${}^V id_U = id_{V \otimes U}$ and ${}^V(g \circ f) = {}^V g \circ {}^V f$. Since the conjugation $\varphi : G \rightarrow Aut(\mathcal{C})$ is a group homomorphism, for all $V, W \in \mathcal{C}$, we have ${}^{V \otimes W}(\cdot) = {}^V({}^W(\cdot))$ and $\mathbb{I}(\cdot) = {}^V(\overline{V}(\cdot)) = \overline{V}({}^V(\cdot)) = id_{\mathcal{C}}$. Since, for all $V \in \mathcal{C}$, the functor ${}^V(\cdot)$ is strict, we have ${}^V(f \otimes g) = {}^V f \otimes {}^V g$, for any morphisms f and g in \mathcal{C} , and ${}^V \mathbb{I} = \mathbb{I}$.

A *braiding* of a crossed category \mathcal{C} is a family of isomorphisms $(c = c_{U,V})_{U,V \in \mathcal{C}}$, where $c_{U,V} : U \otimes V \rightarrow {}^U V \otimes U$ satisfying the following conditions:

(i) For any arrow $f \in \mathcal{C}_\alpha(U, U')$ and $g \in \mathcal{C}(V, V')$,

$$(({}^\alpha g) \otimes f) \circ c_{U,V} = c_{U'V'} \circ (f \otimes g).$$

(ii) For all $U, V, W \in \mathcal{C}$, we have

$$c_{U \otimes V, W} = a_{U \otimes V, W, U, V} \circ (c_{U, V \otimes W} \otimes id_V) \circ a_{U, V, W, V}^{-1} \circ (\iota_U \otimes c_{V, W}) \circ a_{U, V, W},$$

$$c_{U, V \otimes W} = a_{U, V, U, W}^{-1} \circ (\iota_{(U \otimes V)} \otimes c_{U, W}) \circ a_{U, V, U, W} \circ (c_{U, V} \otimes \iota_W) \circ a_{U, V, W}^{-1},$$

where a is the natural isomorphisms in the tensor category \mathcal{C} .

(iii) For all $U, V \in \mathcal{C}$ and $\beta \in G$,

$$\varphi_\beta(c_{U,V}) = c_{\varphi_\beta(U), \varphi_\beta(V)}.$$

A crossed category endowed with a braiding is called a *braided T -category*.

2. Weak monoidal Hom-bialgebras and examples

Let k be a commutative ring. The results of Hom-construction by Stef and Isar[13] can be applied to the category of k -modules (vector spaces if k is a field) $\mathcal{C} = \mathcal{M}_k$.

In this section we will introduce the notion of a weak monoidal Hom-bialgebra.

Definition 2.1. $H = (H, \xi, m, 1_H, \Delta, \varepsilon)$ is called a *weak monoidal Hom-bialgebra* if (H, ξ) is both a monoidal Hom-algebra and a monoidal Hom-coalgebra, satisfying the following identities for any $a, b, c \in H$:

- (1) $\Delta(ab) = \Delta(a)\Delta(b)$;
- (2) $\varepsilon((ab)c) = \varepsilon(ab_1)\varepsilon(b_2c)$, $\varepsilon(a(bc)) = \varepsilon(ab_2)\varepsilon(b_1c)$;
- (3) $(\Delta \otimes id_H)\Delta(1_H) = 1_1 \otimes 1_2 1'_1 \otimes 1'_2$, $(id_H \otimes \Delta)\Delta(1_H) = 1_1 \otimes 1'_1 1_2 \otimes 1'_2$.

Definition 2.2. If (H, ξ) and (H', ξ') are two weak monoidal Hom-bialgebras, a linear map $f : H \rightarrow H'$ is called a *morphism of weak monoidal Hom-bialgebras* if f is both a morphism of monoidal Hom-algebras and a morphism of monoidal Hom-coalgebras.

Let H be a weak monoidal Hom-bialgebra. Define linear maps ε_s and ε_t by the formulas

$$\varepsilon_s(h) = \xi^2(1_1)\varepsilon(\xi^{-2}(h)1_2), \quad \varepsilon_t(h) = \varepsilon(1_1\xi^{-2}(h))\xi^2(1_2),$$

for any $h \in H$, where $\varepsilon_t, \varepsilon_s$ are called the *target* and *source counital maps*. We adopt the notations $H_t = \varepsilon_t(H)$ and $H_s = \varepsilon_s(H)$ for their images.

Similarly, we define the linear maps $\widehat{\varepsilon}_s$ and $\widehat{\varepsilon}_t$ by the formulas

$$\widehat{\varepsilon}_s(h) = \xi^2(1_1)\varepsilon(1_2\xi^{-2}(h)), \quad \widehat{\varepsilon}_t(h) = \varepsilon(\xi^{-2}(h)1_1)\xi^2(1_2),$$

for any $h \in H$. Their images are denoted by $\widehat{H}_t = \widehat{\varepsilon}_t(H)$ and $\widehat{H}_s = \widehat{\varepsilon}_s(H)$. And then we obtain the following identities

$$\Delta(1_H) \in H_s \otimes H_t, \quad \Delta(1_H) \in \widehat{H}_s \otimes \widehat{H}_t,$$

and

$$1_1 \otimes 1_2 = \xi(1_1) \otimes \xi(1_2), \quad \xi(h_1) \otimes \xi(h_2) = 1_1 h_1 \otimes 1_2 h_2 = h_1 1_1 \otimes h_2 1_2.$$

Proposition 2.3. Let H be a weak monoidal Hom-bialgebra. Then for any $h \in H$, we have

$$(i) \quad \varepsilon_s(h_1) \otimes h_2 = \xi^3(1_1) \otimes \xi^{-2}(h)1_2; \quad (2.1)$$

$$(ii) \quad \widehat{\varepsilon}_s(h_1) \otimes h_2 = \xi(1_1) \otimes 1_2 \xi^{-2}(h). \quad (2.2)$$

Proof. (i). From the definition of ε_s , we immediately get that

$$\begin{aligned} \varepsilon_s(h_1) \otimes h_2 &= \xi^2(1_1) \varepsilon(\xi^{-2}(h_1)1_2) \otimes h_2 \\ &= \xi^2(1_1) \varepsilon(\xi^{-2}(h_1)(1'_1 \xi^{-1}(1_2))) \otimes \xi^{-1}(h_2)1'_2 \\ &= \xi^2(1_1) \otimes \xi^{-2}(h) \varepsilon(1'_1 1_2)1'_2 \\ &= \xi^2(1_1) \otimes \xi^{-2}(h) \xi^{-1}(1_2) \\ &= \xi^3(1_1) \otimes \xi^{-2}(h)1_2. \end{aligned}$$

The proof of (ii) is similar to (i). ■

Theorem 2.4. Let H be a weak Hom-bialgebra. Then for any $a, b, c \in H$, we have the following identities

$$(i) \quad \Delta(1_1) \otimes 1_2 = 1_1 \otimes \Delta(1_2); \quad (2.3)$$

$$(ii) \quad \varepsilon((ab)c) = \varepsilon(a(bc)). \quad (2.4)$$

Proof. From the Proposition 2.3 above, we know that $\varepsilon_s(1_1) \otimes 1_2 = \xi^3(1_1) \otimes \xi(1_2)$, another side,

$$\begin{aligned} \varepsilon_s(1_1) \otimes 1_2 &= \xi^2(1'_1) \varepsilon(1_1 1'_2) \otimes \xi^2(1_2) \\ &= \xi^2(1_1) \otimes \xi(1_2) \end{aligned}$$

then we can get

$$1_1 \otimes 1_2 = \xi(1_1) \otimes 1_2. \quad (2.5)$$

Similarly, we can get $\widehat{\varepsilon}_s(1_1) \otimes 1_2 = \xi(1_1) \otimes \xi(1_2)$ from (2.2) and $\widehat{\varepsilon}_s(1_1) \otimes 1_2 = \xi(1_1) \otimes \xi^2(1_2)$ by a direct computation, which means

$$1_1 \otimes 1_2 = 1_1 \otimes \xi(1_2). \quad (2.6)$$

(i). We compute as follows:

$$\Delta(1_1) \otimes 1_2 \stackrel{(2.6)}{=} \Delta(1_1) \otimes \xi^{-1}(1_2) = \xi^{-1}(1_1) \otimes \Delta(1_2) \stackrel{(2.5)}{=} 1_1 \otimes \Delta(1_2).$$

(ii). Since

$$\varepsilon(\xi(a)b) = \varepsilon(a1_1) \varepsilon(1_2 b) = \varepsilon(a1_1) \varepsilon(\xi(1_2) \xi(b)) \stackrel{(2.6)}{=} \varepsilon(a1_1) \varepsilon(1_2 \xi(b)) = \varepsilon(\xi(a) \xi(b)) = \varepsilon(ab),$$

we get

$$\varepsilon(\xi(a)b) = \varepsilon(ab), \quad (2.7)$$

and similarly, we can get

$$\varepsilon(a\xi(b)) = \varepsilon(ab). \quad (2.8)$$

Thus we have

$$\varepsilon(a(bc)) \stackrel{(2.7)}{=} \varepsilon(\xi(a)(bc)) = \varepsilon((ab)\xi(c)) \stackrel{(2.8)}{=} \varepsilon((ab)c).$$

■

So from now on, we can denote $(\Delta \otimes id_H)\Delta(1_H)$ by $1_1 \otimes 1_2 \otimes 1_3$, and $\varepsilon((ab)c)$ by $\varepsilon(abc)$.

From (2.5) and (2.6), we can rewrite the definitions of ε_s , ε_t , $\widehat{\varepsilon}_s$ and $\widehat{\varepsilon}_t$ by the following formulas:

$$\varepsilon_s(h) = 1_1\varepsilon(h1_2) \quad \varepsilon_t(h) = \varepsilon(1_1h)1_2, \quad (2.9)$$

$$\widehat{\varepsilon}_s(h) = 1_1\varepsilon(1_2h) \quad \widehat{\varepsilon}_t(h) = \varepsilon(h1_1)1_2 \quad (2.10)$$

Corollary 2.5. Let H be a weak monoidal Hom-bialgebra. Then for any $a, b, c \in H$, we have

$$(i) \quad \varepsilon_s(ab) = \varepsilon_s(\xi(a)b) = \varepsilon_s(a\xi(b)), \quad \varepsilon_t(ab) = \varepsilon_t(\xi(a)b) = \varepsilon_t(a\xi(b)); \quad (2.11)$$

$$\widehat{\varepsilon}_s(ab) = \widehat{\varepsilon}_s(\xi(a)b) = \widehat{\varepsilon}_s(a\xi(b)), \quad \widehat{\varepsilon}_t(ab) = \widehat{\varepsilon}_t(\xi(a)b) = \widehat{\varepsilon}_t(a\xi(b)); \quad (2.12)$$

$$(ii) \quad \varepsilon_s \circ \varepsilon_s = \varepsilon_s \circ \xi = \xi \circ \varepsilon_s = \varepsilon_s, \quad \varepsilon_t \circ \varepsilon_t = \varepsilon_t \circ \xi = \xi \circ \varepsilon_t = \varepsilon_t; \quad (2.13)$$

$$(iii) \quad \widehat{\varepsilon}_s \circ \widehat{\varepsilon}_s = \widehat{\varepsilon}_s \circ \xi = \xi \circ \widehat{\varepsilon}_s = \widehat{\varepsilon}_s, \quad \widehat{\varepsilon}_t \circ \widehat{\varepsilon}_t = \widehat{\varepsilon}_t \circ \xi = \xi \circ \widehat{\varepsilon}_t = \widehat{\varepsilon}_t. \quad (2.14)$$

Proof. Using equations (2.5) to (2.10), left to readers.

Proposition 2.6. Let H be a weak monoidal Hom-bialgebra. Then for any $x, y, h \in H$, we have

$$(i) \quad \varepsilon(xy) = \varepsilon(\varepsilon_s(x)y) = \varepsilon(x\varepsilon_t(y)); \quad (2.15)$$

$$\varepsilon(xy) = \varepsilon(\widehat{\varepsilon}_t(x)y) = \varepsilon(x\widehat{\varepsilon}_s(y)); \quad (2.16)$$

$$(ii) \quad x_1 \otimes \varepsilon_t(x_2) = 1_1\xi^{-2}(x) \otimes 1_2, \quad \varepsilon_s(x_1) \otimes x_2 = 1_1 \otimes \xi^{-2}(x)1_2; \quad (2.17)$$

$$\widehat{\varepsilon}_s(x_1) \otimes x_2 = 1_1 \otimes 1_2\xi^{-2}(x), \quad x_1 \otimes \widehat{\varepsilon}_t(x_2) = \xi^{-2}(x)1_1 \otimes 1_2; \quad (2.18)$$

$$(iii) \quad x_1\varepsilon(hx_2) = \varepsilon_s(h)\xi^{-2}(x), \quad \varepsilon(x_1h)x_2 = \xi^{-2}(x)\varepsilon_t(h); \quad (2.19)$$

$$\varepsilon(hx_1)x_2 = \widehat{\varepsilon}_t(h)\xi^{-2}(x), \quad x_1\varepsilon(x_2h) = \xi^{-2}(x)\widehat{\varepsilon}_s(h); \quad (2.20)$$

$$(iv) \quad \varepsilon_s(x)\varepsilon_t(y) = \varepsilon_t(y)\varepsilon_s(x), \quad \widehat{\varepsilon}_s(x)\widehat{\varepsilon}_t(y) = \widehat{\varepsilon}_t(y)\widehat{\varepsilon}_s(x). \quad (2.21)$$

Proof. We just check some of them and the others are left to readers.

(i).

$$\varepsilon(\varepsilon_s(x)y) \stackrel{(2.9)}{=} \varepsilon(1_1y)\varepsilon(x1_2) = \varepsilon(x\xi(y)) \stackrel{(2.8)}{=} \varepsilon(xy).$$

(ii). Similar to Proposition 2.3.

(iii).

$$x_1\varepsilon(hx_2) \stackrel{(2.15)}{=} x_1\varepsilon(h\varepsilon_t(x_2)) = 1_1\xi^{-2}(x)\varepsilon(h1_2) = \varepsilon_s(h)\xi^{-2}(x).$$

(iv).

$$\varepsilon_s(x)\varepsilon_t(y) = 1_1\varepsilon_t(y)\varepsilon(x1_2) \stackrel{(2.19)}{=} \varepsilon(1_1y)1_2\varepsilon(x1_3) = \varepsilon_t(y)\varepsilon_s(x).$$

■

Proposition 2.7. Let H be a weak monoidal Hom-bialgebra. Then for any $h \in H$, we have

$$\Delta(\varepsilon_t(h)) = 1_1\varepsilon_t(h) \otimes 1_2 = \varepsilon_t(h)1_1 \otimes 1_2; \quad (2.22)$$

$$\Delta(\varepsilon_s(h)) = 1_1 \otimes 1_2\varepsilon_s(h) = 1_1 \otimes \varepsilon_s(h)1_2; \quad (2.23)$$

$$\Delta(\widehat{\varepsilon}_s(h)) = 1_1 \otimes \widehat{\varepsilon}_s(h)1_2 = 1_1 \otimes 1_2\widehat{\varepsilon}_s(h); \quad (2.24)$$

$$\Delta(\widehat{\varepsilon}_t(h)) = 1_1\widehat{\varepsilon}_t(h) \otimes 1_2 = \widehat{\varepsilon}_t(h)1_1 \otimes 1_2. \quad (2.25)$$

Proof. We only check Eq.(2.22). Indeed,

$$\Delta(\varepsilon_t(h)) = \varepsilon(1_1h)1_2 \otimes 1_3 \stackrel{(2.19)}{=} 1_1\varepsilon_t(h) \otimes 1_2 \stackrel{(2.21)}{=} \varepsilon_t(h)1_1 \otimes 1_2.$$

The other three identities can be proved by similar calculations. ■

Proposition 2.8. Let H be a weak monoidal Hom-bialgebra. Then for any $x, y \in H$, we have

$$(i) \quad \varepsilon_t(x\varepsilon_t(y)) = \varepsilon_t(xy), \quad \varepsilon_t(\varepsilon_t(x)y) = \varepsilon_t(x)\varepsilon_t(y); \quad (2.26)$$

$$\varepsilon_s(x\varepsilon_s(y)) = \varepsilon_s(x)\varepsilon_s(y), \quad \varepsilon_s(\varepsilon_s(x)y) = \varepsilon_s(xy); \quad (2.27)$$

$$(ii) \quad \widehat{\varepsilon}_t(x\widehat{\varepsilon}_t(y)) = \widehat{\varepsilon}_t(x)\widehat{\varepsilon}_t(y), \quad \widehat{\varepsilon}_t(\widehat{\varepsilon}_t(x)y) = \widehat{\varepsilon}_t(xy); \quad (2.28)$$

$$\widehat{\varepsilon}_s(x\widehat{\varepsilon}_s(y)) = \widehat{\varepsilon}_s(xy), \quad \widehat{\varepsilon}_s(\widehat{\varepsilon}_s(x)y) = \widehat{\varepsilon}_s(x)\widehat{\varepsilon}_s(y); \quad (2.29)$$

$$(iii) \quad \varepsilon_t(x\widehat{\varepsilon}_s(y)) = \varepsilon_t(xy), \quad \varepsilon_t(\widehat{\varepsilon}_t(x)y) = \widehat{\varepsilon}_t(x)\varepsilon_t(y); \quad (2.30)$$

$$\varepsilon_s(x\widehat{\varepsilon}_t(y)) = \varepsilon_s(x)\widehat{\varepsilon}_t(y), \quad \varepsilon_s(\widehat{\varepsilon}_t(x)y) = \varepsilon_s(xy); \quad (2.31)$$

$$(iv) \quad \widehat{\varepsilon}_t(x\varepsilon_t(y)) = \widehat{\varepsilon}_t(x)\varepsilon_t(y), \quad \widehat{\varepsilon}_t(\varepsilon_s(x)y) = \widehat{\varepsilon}_t(xy); \quad (2.32)$$

$$\widehat{\varepsilon}_s(x\varepsilon_t(y)) = \widehat{\varepsilon}_s(xy), \quad \widehat{\varepsilon}_s(\varepsilon_s(x)y) = \varepsilon_s(x)\widehat{\varepsilon}_s(y); \quad (2.33)$$

$$(v) \quad \varepsilon_t(x_1) \otimes x_2 = \varepsilon_t(1_1) \otimes 1_2\xi^{-2}(x), \quad x_1 \otimes \varepsilon_s(x_2) = \xi^{-2}(x)1_1 \otimes \varepsilon_s(1_2); \quad (2.34)$$

$$x_1 \otimes \widehat{\varepsilon}_s(x_2) = 1_1\xi^{-2}(x) \otimes \widehat{\varepsilon}_s(1_2), \quad \widehat{\varepsilon}_t(x_1) \otimes x_2 = \widehat{\varepsilon}_t(1_1) \otimes \xi^{-2}(x)1_2. \quad (2.35)$$

Proof. Straightforward.

Remark. For any weak monoidal Hom-bialgebra (H, ξ) , if $\xi = id_H$, then H is exactly a weak bialgebra. If Δ and ε are all monoidal Hom-algebra morphisms, then H is exactly a monoidal Hom-bialgebra.

Hereby we give examples of weak monoidal Hom-bialgebras.

Example 2.9. If $(H, \xi, m, \eta, \Delta, \varepsilon)$ is a finite dimensional weak monoidal Hom-bialgebra and H^* is the linear dual of H . Then $(H^*, \xi^*, m^*, \eta^*, \Delta^*, \varepsilon^*)$ is also a finite dimensional weak monoidal Hom-bialgebra, where

$$\xi^*(f) = f \circ \xi^{-1},$$

for any $f \in H^*$.

Example 2.10. (10-dimensional weak monoidal Hom-bialgebra) Let H be a vector space over k with basis $\{x_i\}$, where $i = 1, 2, \dots, 10$ and let $0 \neq \lambda \in k$, we give the structure of H as follows:

- the multiplication

H	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	x_1	x_2	x_3	λx_4	λx_5	x_6	x_7	x_8	λx_9	λx_{10}
x_2	x_2	x_2	x_3	λx_4	λx_5	x_7	x_7	x_8	λx_9	λx_{10}
x_3	x_3	x_3	x_2	$-\lambda x_5$	$-\lambda x_4$	x_8	x_8	x_7	$-\lambda x_{10}$	$-\lambda x_9$
x_4	λx_4	λx_4	λx_5	0	0	λx_9	λx_9	λx_{10}	0	0
x_5	λx_5	λx_5	λx_4	0	0	λx_{10}	λx_{10}	λx_9	0	0
x_6	x_6	x_7	x_8	λx_9	λx_{10}	x_6	x_7	x_8	λx_9	λx_{10}
x_7	x_7	x_7	x_8	λx_9	λx_{10}	x_7	x_7	x_8	λx_9	λx_{10}
x_8	x_8	x_8	x_7	$-\lambda x_{10}$	$-\lambda x_9$	x_8	x_8	x_7	$-\lambda x_{10}$	$-\lambda x_9$
x_9	λx_9	λx_9	λx_{10}	0	0	λx_9	λx_9	λx_{10}	0	0
x_{10}	λx_{10}	λx_{10}	λx_9	0	0	λx_{10}	λx_{10}	λx_9	0	0

- the comultiplication

$$\begin{aligned}
\Delta(x_1) &= x_1 \otimes x_1 - x_1 \otimes x_6 - x_6 \otimes x_1 + 2x_6 \otimes x_6 - x_1 \otimes x_2 + x_1 \otimes x_7 + x_6 \otimes x_2 \\
&\quad - 2x_6 \otimes x_7 - x_2 \otimes x_1 + x_2 \otimes x_6 + x_7 \otimes x_1 - 2x_7 \otimes x_6 + 2x_2 \otimes x_2 - 2x_2 \otimes x_7 - 2x_7 \otimes x_2 + 4x_7 \otimes x_7, \\
\Delta(x_2) &= x_2 \otimes x_2 - x_2 \otimes x_7 - x_7 \otimes x_2 + 2x_7 \otimes x_7, \\
\Delta(x_3) &= x_3 \otimes x_3 - x_3 \otimes x_8 - x_8 \otimes x_3 + 2x_8 \otimes x_8, \\
\Delta(x_4) &= \frac{1}{\lambda}(x_3 \otimes x_4 - x_3 \otimes x_9 - x_8 \otimes x_4 + 2x_8 \otimes x_9 + x_4 \otimes x_2 - x_4 \otimes x_7 - x_9 \otimes x_2 + 2x_9 \otimes x_7), \\
\Delta(x_5) &= \frac{1}{\lambda}(x_2 \otimes x_5 - x_2 \otimes x_{10} - x_7 \otimes x_5 + 2x_7 \otimes x_{10} + x_5 \otimes x_3 - x_5 \otimes x_8 - x_{10} \otimes x_8 + 2x_{10} \otimes x_8), \\
\Delta(x_6) &= x_6 \otimes x_6 - x_6 \otimes x_7 - x_7 \otimes x_6 + 2x_7 \otimes x_7 \\
\Delta(x_7) &= x_7 \otimes x_7, \\
\Delta(x_8) &= x_8 \otimes x_8, \\
\Delta(x_9) &= \frac{1}{\lambda}(x_8 \otimes x_9 + x_9 \otimes x_7), \\
\Delta(x_{10}) &= \frac{1}{\lambda}(x_7 \otimes x_{10} + x_{10} \otimes x_8);
\end{aligned}$$

- the counit

$$\begin{aligned}
\varepsilon(x_1) &= 4, \varepsilon(x_6) = 2, \varepsilon(x_2) = \varepsilon(x_3) = 2, \varepsilon(x_7) = \varepsilon(x_8) = 1, \\
\varepsilon(x_4) &= \varepsilon(x_5) = \varepsilon(x_9) = \varepsilon(x_{10}) = 0;
\end{aligned}$$

Then we define a map $\xi : H \rightarrow H$. Then it is an isomorphism of weak monoidal Hom-bialgebra if its matrix of the basis $\{x_i | i = 1, 2, \dots, 10\}$ takes the form

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

Then it is a direct computation to check that (H, ξ) is a weak monoidal Hom-bialgebra.

Definition 2.11. A weak monoidal Hom-bialgebra (H, ξ) is called a *weak monoidal Hom-Hopf algebra* if H endowed with a k -linear map S (the *antipode*), such that for any $h, g \in H$, the following conditions hold:

- (1) $S \circ \xi = \xi \circ S$;
- (2) $h_1 S(h_2) = \varepsilon_t(h)$, $S(h_1) h_2 = \varepsilon_s(h)$;
- (3) $S(hg) = S(g)S(h)$, $S(1_H) = 1_H$;
- (4) $\Delta(S(h)) = S(h_2) \otimes S(h_1)$, $\varepsilon \circ S = \varepsilon$.

Proposition 2.12. H is a weak monoidal Hom-Hopf algebra, then for any $h \in H$, the following equalities hold:

$$(i) \quad \varepsilon_s(h_1)S(h_2) = \xi^{-1}(S(h)), \quad S(h_1)\varepsilon_t(h_2) = \xi^{-1}(S(h)); \quad (2.36)$$

$$(ii) \quad \varepsilon_t(h) = S(\widehat{\varepsilon_s}(h)), \quad \varepsilon_s(h) = S(\widehat{\varepsilon_t}(h)); \quad (2.37)$$

$$(iii) \quad \varepsilon_t(h_1) \otimes h_2 = S(1_1) \otimes 1_2 \xi^{-2}(h), \quad h_1 \otimes \varepsilon_s(h_2) = \xi^{-2}(h) 1_1 \otimes S(1_2). \quad (2.38)$$

Proof. We only check the first identities of each one.

(i).

$$\begin{aligned} \varepsilon_s(h_1)S(h_2) &= 1_1 \varepsilon(h_1 1'_1 1_2) S(\xi^{-1}(h_2) 1'_2) \\ &= 1_1 S(\xi^{-2}(h) 1_2) = \xi^{-2}(S(h)) 1 = \xi^{-1}(S(h)); \end{aligned}$$

(ii)

$$\begin{aligned} \varepsilon_t(h) &= \varepsilon(1_1 \varepsilon_t(h)) 1_2 \stackrel{(2.21)}{=} \varepsilon(\varepsilon_t(h) 1_1) 1_2 \\ &\stackrel{(2.15)}{=} \varepsilon(\varepsilon_s(h_1) S(h_2) 1_1) 1_2 = \varepsilon(S(h) 1_1) 1_2 \\ &= \varepsilon(S(h) S(1_2)) S(1_1) = S(1_1) \varepsilon(1_2 h) = S(\widehat{\varepsilon_s}(h)); \end{aligned}$$

(iii) Easy to get from (ii). ■

Note that, a monoidal Hom-bialgebra is Hom-bialgebra if and only if the Hom-structure map ξ satisfies $\xi \circ \xi = id$. Moreover, through a direct computation, we can get that there is a one to one correspondence between the collection the monoidal Hom-bialgebras over a commutative ring k , and the collection of the unital Hom-bialgebra over k which Hom-structure map is a bijection.

Recall from [19], a weak Hom-bialgebra is a Hom-algebra and a Hom-coalgebra with the compatible conditions as follows:

- (1) $\Delta(ab) = \Delta(a)\Delta(b)$;
- (2) $\varepsilon((ab)c) = \varepsilon(ab_1)\varepsilon(b_2c)$, $\varepsilon(a(bc)) = \varepsilon(ab_2)\varepsilon(b_1c)$;
- (3) $(\Delta \otimes id_H)\Delta(1_H) = 1_1 \otimes 1_2 1'_1 \otimes 1'_2$, $(id_H \otimes \Delta)\Delta(1_H) = 1_1 \otimes 1'_1 1_2 \otimes 1'_2$.

More precisely, we can obtain the following relationship between the collection of weak monoidal Hom-Hopf algebras and weak Hom-Hopf algebras.

Proposition 2.13. If $(H, \xi, m, 1_H, \Delta, \varepsilon, S)$ is a weak monoidal Hom-Hopf algebra, then ${}^\xi H = (H, \xi, m, 1_H, \Delta \circ \xi^2, \varepsilon, S)$ is a weak Hom-Hopf algebra. Conversely, if $(B, \xi_B, m, 1_B, \Delta, \varepsilon, S)$ is a weak Hom-Hopf algebra and ξ_B is invertible, then ${}_\xi B = (B, \xi_B, m, 1_B, \Delta \circ \xi_B^{-2}, \varepsilon, S)$ is a weak monoidal Hom-Hopf algebra.

Proof. Firstly we denote $\Delta \circ \xi^2(h)$ by $h_{[1]} \otimes h_{[2]}$. Then we have

$$\begin{aligned} \xi(h_{[1]}) \otimes h_{[2][1]} \otimes h_{[2][2]} &= \xi(h_{[1]}) \otimes \xi^2(h_{[2]1}) \otimes \xi^2(h_{[2]2}) \\ &= \xi^3(h_1) \otimes \xi^4(h_{21}) \otimes \xi^4(h_{22}) \\ &= \xi^4(h_{11}) \otimes \xi^4(h_{12}) \otimes \xi^3(h_2) \\ &= h_{[1][1]} \otimes h_{[1][2]} \otimes \xi(h_{[2]}), \end{aligned}$$

and

$$\begin{aligned} \varepsilon(h_{[1]})h_{[2]} &= \varepsilon(h_1)\xi^2(h_2) \\ &= \xi(h) \\ &= h_{[1]}\varepsilon(h_{[2]}), \end{aligned}$$

which implies $({}^\xi H, \xi)$ is a Hom-coalgebra. Obviously $({}^\xi H, \xi)$ is a Hom-algebra.

Secondly, we have

$$\varepsilon((xy)z) = \varepsilon(xy_1)\varepsilon(y_2z) = \varepsilon(xy_{[1]})\varepsilon(y_{[2]}z),$$

and the other conditions are easily to get. The proof of the opposite statement is left to readers. ■

Proposition 2.14. We find that ([19], Proposition 2.9) should be like:

For any given weak bialgebra $(H, \mu, 1_H, \Delta, \varepsilon)$, suppose that $\alpha : H \rightarrow H$ is both a morphism of algebras preserving unit and a morphism of coalgebras preserving counit. Thus we can define a new multiplication $\bar{\mu} := \alpha \circ \mu$, and a new comultiplication $\bar{\Delta} := \Delta \circ \alpha$, then $H^\alpha = (H, \alpha, \bar{\mu}, \eta, \bar{\Delta}, \varepsilon)$ is a weak Hom-bialgebra if and only if α satisfies $\alpha(1_1) \otimes 1_2 = \Delta(1_H)$.

Proof. \Leftarrow : We denote $\bar{\mu}(a \otimes b)$ by $a \circ b$, $\bar{\Delta}(c) = c_{[1]} \otimes c_{[2]}$ for any $a, b, c \in H$.

Firstly, we have

$$\begin{aligned} (a \circ b) \circ \alpha(c) &= \alpha(\alpha(a)\alpha(b))\alpha^2(c) \\ &= \alpha(\alpha(a)\alpha(bc)) = \alpha(a) \circ (b \circ c). \end{aligned}$$

Since $\alpha(1_H) = 1_H$, thus $(H, \alpha, \bar{\mu}, 1_H)$ is a Hom-algebra.

Similarly, $(H, \alpha, \bar{\Delta}, \varepsilon)$ is a Hom-coalgebra.

Secondly, from $\alpha(1_1) \otimes 1_2 = \Delta(1_H)$, we have $1_1 \otimes \alpha(1_2) = \Delta(1_H)$. Thus we can get $1_{[1]} \otimes 1_{[2][1]} \otimes 1_{[2][2]} = 1_{[1]} \otimes 1'_{[1]}1_{[2]} \otimes 1'_{[2]}$ and $1_{[1][1]} \otimes 1_{[1][2]} \otimes 1_{[2]} = 1_{[1]} \otimes 1_{[2]}1'_{[1]} \otimes 1'_{[2]}$.

Thirdly, since $\alpha(1_1) \otimes 1_2 = \Delta(1_H)$, we immediately get

$$\begin{aligned} \varepsilon(\alpha(a)b) &= \varepsilon(\alpha(a)1_1)\varepsilon(1_2b) \\ &= \varepsilon(\alpha(a)\alpha(1_1))\varepsilon(1_2b) = \varepsilon(a1_1)\varepsilon(1_2b) \\ &= \varepsilon(ab), \end{aligned}$$

then we have

$$\begin{aligned}
\varepsilon((a \circ b) \circ c) &= \varepsilon(\alpha(\alpha(ab)c)) = \varepsilon(\alpha^2(a)\alpha^2(b_1))\varepsilon(\alpha^2(b_2)\alpha(c)) \\
&= \varepsilon(\alpha(a)\alpha^2(b_1))\varepsilon(\alpha(\alpha(b_2)_2c)) \\
&= \varepsilon(a \circ b_{[1]})\varepsilon(b_{[2]} \circ c),
\end{aligned}$$

and similarly we can obtain $\varepsilon(a\alpha(b)) = \varepsilon(ab)$ through $1_1 \otimes \alpha(1_2) = \Delta(1_H)$. Thus we have $\varepsilon(a \circ (b \circ c)) = \varepsilon(a \circ b_{[2]})\varepsilon(b_{[1]} \circ c)$.

Finally, we check that

$$\begin{aligned}
(a \circ b)_{[1]} \otimes (a \circ b)_{[2]} &= \alpha(\alpha(a)_1\alpha(b)_1) \otimes \alpha(\alpha(a)_2\alpha(b)_2) \\
&= a_{[1]} \circ b_{[1]} \otimes a_{[2]} \circ b_{[2]},
\end{aligned}$$

which means that $H^\alpha = (H, \alpha, \overline{\mu}, 1_H, \Delta, \varepsilon)$ is a weak Hom-bialgebra.

\Rightarrow : Straightforward. ■

Remark. Based on the above proposition, it is easy to know that every 2-dimensional weak Hom-bialgebras have trivial structures. That means if H_2 is a \mathbb{k} -space with a basis I, E , and the following structures

- the multiplication

H_2	I	E
I	I	E
E	E	E

- the comultiplication

$$\Delta(I) = (I - E) \otimes (I - E) + E \otimes E, \quad \Delta(E) = E \otimes E;$$

- the counit

$$\varepsilon(I) = 2, \quad \varepsilon(E) = 1.$$

Then the automorphism of H_2 is identity map. This means ([19], Example 2.12) is a trivial weak bialgebra.

3. Weak (α, β) - Yetter-Drinfeld monoidal Hom-modules

In this section, we will define the notion of a Yetter-Drinfeld module over a weak monoidal Hom-Hopf algebra that is twisted by two weak monoidal Hom-Hopf algebra automorphisms as well as the notion of a weak monoidal Hom-entwining structure and how to obtain such structure from automorphisms of weak monoidal Hom-Hopf algebras.

In what follows, let (H, ξ) be a weak monoidal Hom-Hopf algebra with the bijective antipode S and let $Aut_{wmHH}(H)$ denote the set of all automorphisms of a weak monoidal Hopf algebra H .

Definition 3.1. Let $\alpha, \beta \in \text{Aut}_{wmHH}(H)$. A weak left-right (α, β) -Yetter-Drinfeld Hom-module over (H, ξ) is a vector space M such that:

- (1) (M, \cdot, μ) is a left H -Hom-module;
- (2) (M, ρ, μ) is a right H -Hom-comodule;
- (3) ρ and \cdot satisfy the following compatibility condition:

$$\rho(h \cdot m) = \xi(h_{21}) \cdot m_{(0)} \otimes (\beta(h_{22})\xi^{-1}(m_{(1)}))\alpha(S^{-1}(h_1)), \quad (3.1)$$

for all $h \in H$ and $m \in M$. We denote by ${}_H\mathcal{WMHYD}^H(\alpha, \beta)$ the category of weak left-right (α, β) -Yetter-Drinfeld Hom-modules, morphisms being H -linear H -colinear maps.

Remark. Note that, α and β are bijective, Hom-algebra morphisms, Hom-coalgebra morphisms, and commute with S and ξ .

Proposition 3.2. One has that Eq.(3.1) is equivalent to the following equations:

$$\rho(m) = m_{(0)} \otimes m_{(1)} \in M \otimes_{t_\beta} H \stackrel{\Delta}{=} (1_1 \otimes \beta(1_2)) \cdot (M \otimes H), \forall m \in M, \quad (3.2)$$

$$h_1 \cdot m_{(0)} \otimes \beta(h_2)m_{(1)} = \mu((h_2 \cdot \mu^{-1}(m))_{(0)}) \otimes (h_2 \cdot \mu^{-1}(m))_{(1)}\alpha(h_1). \quad (3.3)$$

Proof. Eq.(3.1) \implies Eq.(3.2, 3.3). We first note that

$$\begin{aligned} m_{(0)} \otimes m_{(1)} &= \xi(1_{21}) \cdot \mu^{-1}(m)_{(0)} \otimes (\beta(1_{22})\xi^{-2}(m_{(1)}))\alpha(S^{-1}(1_1)) \\ &= 1_{21} \cdot \mu^{-1}(m)_{(0)} \otimes ((\beta(1_{22})\xi^{-2}(m_{(1)}))\alpha(S^{-1}(1_1))) \\ &= 1'_1 \cdot (1_2 \cdot \mu^{-2}(m)_{(0)}) \otimes \beta(1'_{(2)})(\xi^{-2}(m_{(1)})\alpha(S^{-1}(1_1))) \in M \otimes_{t_\beta} H. \end{aligned}$$

Then we do calculation as follows:

$$\begin{aligned} &\mu((h_2 \cdot \mu^{-1}(m))_{(0)}) \otimes (h_2 \cdot \mu^{-1}(m))_{(1)}\alpha(h_1) \\ \stackrel{(3.1)}{=} &\mu(\xi(h_{221}) \cdot \mu^{-1}(m)_{(0)}) \otimes ((\beta(h_{222})\xi^{-2}(m_{(1)}))\alpha(S^{-1}(h_{21})))\alpha(h_1) \\ = &\xi^{-1}(h_1 1_2) \cdot m_{(0)} \otimes \xi^{-1}(\beta(h_2)m_{(1)})\alpha(S^{-1}(1_1)) \\ = &h_1 \cdot (1_2 \cdot \mu^{-1}(m_{(0)})) \otimes \xi^{-1}(\beta(h_2)(\beta(1_3)\xi^{-1}(m_{(1)})))\alpha(S^{-1}(1_1)) \\ = &h_1 \cdot m_{(0)} \otimes \beta(h_2)m_{(1)}. \end{aligned}$$

For Eq.(3.2, 3.3) \implies Eq.(3.1), we have

$$\begin{aligned} &\xi(h_{21}) \cdot m_{(0)} \otimes_{t_\beta} (\beta(h_{22})\xi^{-1}(m_{(1)}))\alpha(S^{-1}(h_1)) \\ \stackrel{(3.3)}{=} &\mu((\xi(h_{22}) \cdot \mu^{-1}(m))_{(0)}) \otimes_{t_\beta} \xi^{-1}((\xi(h_{22}) \cdot \mu^{-1}(m))_{(1)}\alpha(\xi(h_{21})))\alpha(S^{-1}(h_1)) \\ = &\mu(((1_2\xi^{-2}(h)) \cdot \mu^{-1}(m))_{(0)}) \otimes_{t_\beta} ((1_2\xi^{-2}(h)) \cdot \mu^{-1}(m))_{(1)}\alpha(1_1) \\ = &\mu((1_2 \cdot \mu^{-1}(\mu^{-1}(h \cdot m)))_{(0)}) \otimes_{t_\beta} (1_2 \cdot \mu^{-1}(\mu^{-1}(h \cdot m)))_{(1)}\alpha(1_1) \\ \stackrel{(3.3)}{=} &1_1 \cdot \mu^{-1}(h \cdot m)_{(0)} \otimes_{t_\beta} \beta(1_2)\xi^{-1}(h \cdot m)_{(1)} \\ \stackrel{(3.2)}{=} &1'_1 \cdot (1_1 \cdot \mu^{-1}(h \cdot m)_{(0)}) \otimes \beta(1'_2) \cdot (\beta(1_2)\xi^{-1}(h \cdot m)_{(1)}) \\ = &(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)}. \end{aligned}$$

This finishes the proof. ■

Definition 3.3. A weak left-right *monoidal Hom-entwining structure* is a triple (H, C, ψ) , where (H, ξ) is a monoidal Hom-algebra and (C, γ) is a monoidal Hom-coalgebra with a linear map $\psi : H \otimes C \rightarrow H \otimes C$, $h \otimes c \mapsto_{\psi} h \otimes c^{\psi}$ satisfying the following conditions:

$$\psi(hg) \otimes c^{\psi} =_{\phi} h_{\psi}g \otimes \gamma(\gamma^{-1}(c)^{\psi\phi}), \quad (3.4)$$

$$_{\psi}1_H \otimes c^{\psi} = \varepsilon(c_1^{\psi})_{\psi}1_H \otimes \gamma(c_2), \quad (3.5)$$

$$_{\psi}h \otimes \Delta(c^{\psi}) = \xi(\phi\psi\xi^{-1}(h)) \otimes (c_1^{\psi} \otimes c_2^{\phi}), \quad (3.6)$$

$$\varepsilon(c^{\psi})_{\psi}h = \varepsilon(\gamma^{-1}(c)^{\psi})h(_{\psi}1_H), \quad (3.7)$$

Over a weak monoidal Hom-entwining structure (H, C, ψ) , a left-right weak monoidal entwined Hom-module M is both a right C -Hom-comodule and a left H -Hom-module such that

$$\rho^M(h \cdot m) = _{\psi}\xi^{-1}(h) \cdot m_{(0)} \otimes \gamma(m_{(1)})^{\psi}$$

for all $h \in H$ and $m \in M$. We denote the category of all monoidal entwined Hom-modules over (H, C, ψ) by ${}_H\mathcal{M}^C(\psi)$.

Let (H, ξ) be a weak monoidal Hom-Hopf algebra with S , and define a linear map

$$\psi(\alpha, \beta) : H \otimes H \rightarrow H \otimes H, \quad a \otimes c \mapsto _{\psi}a \otimes c^{\psi} = \xi^2(a_{21}) \otimes (\beta(a_{22})\xi^{-2}(c))\alpha(S^{-1}(a_1)),$$

for all $\alpha, \beta \in \text{Aut}_{wmHH}(H)$.

Proposition 3.4. With notations above, $(H, H, \psi(\alpha, \beta))$ is a weak monoidal Hom-entwining structure for all $\alpha, \beta \in \text{Aut}_{wmHH}(H)$.

Proof. We need to prove that Eqs.(3.4-3.7) hold. First, it is straightforward to check Eqs.(3.5) and (3.7). In what follows, we only verify Eqs.(3.4) and (3.6). In fact, for all $a, b, c \in H$, we have

$$\begin{aligned} & \phi a_{\psi}b \otimes \xi(\xi^{-1}(c)^{\psi\phi}) \\ &= \xi^2(a_{21})_{\psi}b \otimes \xi((\beta(a_{22})\xi^{-2}(\xi^{-1}(c)^{\psi}))\alpha S^{-1}(a_1)) \\ &= \xi^2(a_{21}b_{21}) \otimes (\beta(a_{22}b_{22})\xi^{-2}(c))(\alpha S^{-1}(b_1)\alpha S^{-1}(a_1)) =_{\psi} (ab) \otimes c^{\psi}, \end{aligned}$$

and Eq.(3.4) is proven.

For all $a \in H$, we have

$$a_1 \otimes a_{211} \otimes a_{2121} \otimes a_{2122} \otimes a_{22} = \xi(a_{11}) \otimes \xi^{-1}(a_{12}) \otimes \xi^{-2}(a_{21}) \otimes \xi^{-1}(a_{221}) \otimes \xi(a_{222}) \quad (3.8)$$

As for Eq.(3.6), we compute:

$$\begin{aligned} & \xi(\phi\psi\xi^{-1}(a)) \otimes (c_1^{\phi} \otimes c_2^{\psi}) \\ &= \xi(\xi^2((_{\psi}\xi^{-1}(a))_{21})) \otimes ((\beta((_{\psi}\xi^{-1}(a))_{22})\xi^{-2}(c_1))\alpha S^{-1}((_{\psi}\xi^{-1}(a))_1) \otimes c_2^{\psi}) \end{aligned}$$

$$\begin{aligned}
&= \xi^2(a_{21}) \otimes ((\beta(a_{221})\xi^{-2}(c_1))\alpha S^{-1}(a_{12}) \otimes (\beta(a_{222})\xi^{-2}(c_2))\alpha S^{-1}(a_{11})) \\
&= {}_\psi a \otimes \Delta(c^\psi).
\end{aligned}$$

and Eq.(3.6) is proven.

This finishes the proof. ■

Remark. By Proposition above, we have a weak monoidal entwined Hom-module category ${}_H\mathcal{M}^H(\psi(\alpha, \beta))$ over $(H, H, \psi(\alpha, \beta))$ with $\alpha, \beta \in \text{Aut}_{wmHH}(H)$. In this case, for all $M \in {}_H\mathcal{M}^H(\psi(\alpha, \beta))$, we have

$$\rho(h \cdot m) = \xi(h_{21}) \cdot m_{(0)} \otimes (\beta(h_{22})\xi^{-1}(m_{(1)}))\alpha(S^{-1}(h_1)),$$

for all $h \in H, m \in M$. Thus means that ${}_H\mathcal{M}^H(\psi(\alpha, \beta)) = {}_H\mathcal{WMHYD}^H(\alpha, \beta)$ as categories.

4. A BRAIDED T -CATEGORY $\mathcal{WMHYD}(H)$

In this section, we will construct a class of new braided T -categories $\mathcal{WMHYD}(H)$ over any weak monoidal Hom-Hopf algebra (H, ξ) with bijective antipode.

Let $(M, \mu) \in {}_H\mathcal{WMHYD}^H(\alpha, \beta), (N, \nu) \in {}_H\mathcal{WMHYD}^H(\gamma, \delta)$, with $\alpha, \beta, \gamma, \delta \in \text{Aut}_{wmHH}(H)$. Define $M \otimes_{t_{\gamma^{-1}\beta}} N = (1_1 \otimes \gamma^{-1}\beta(1_2)) \cdot (M \otimes N)$.

Proposition 4.1. If $(M, \mu) \in {}_H\mathcal{WMHYD}^H(\alpha, \beta)$ and $(N, \nu) \in {}_H\mathcal{WMHYD}^H(\gamma, \delta)$, with $\alpha, \beta, \gamma, \delta \in \text{Aut}_{wmHH}(H)$, then $(M \otimes_{t_{\gamma^{-1}\beta}} N, \mu \otimes \nu) \in {}_H\mathcal{WMHYD}^H(\alpha\gamma, \delta\gamma^{-1}\beta\gamma)$ with structures as follows:

$$\begin{aligned}
h \cdot (m \otimes n) &= \gamma(h_1) \cdot m \otimes \gamma^{-1}\beta\gamma(h_2) \cdot n, \\
m \otimes n &\mapsto (m_{(0)} \otimes n_{(0)}) \otimes n_{(1)}m_{(1)}.
\end{aligned}$$

for all $m \in M, n \in N$ and $h \in H$.

Proof. Let $h, g \in H$ and $m \otimes n \in M \otimes_{t_{\gamma^{-1}\beta}} N$. We can prove $(hg) \cdot (m \otimes n) = \xi(h) \cdot (g \cdot (\mu^{-1}(m) \otimes \nu^{-1}(n)))$ straightforwardly, and

$$\begin{aligned}
1 \cdot (m \otimes_{t_{\gamma^{-1}\beta}} n) &= \gamma(1'_1) \cdot (1_1 \cdot m) \otimes \gamma^{-1}\beta\gamma(1'_2) \cdot (\gamma^{-1}\beta(1_2) \cdot n) \\
&= (\gamma(1'_1)1_1) \cdot \mu(m) \otimes (\gamma^{-1}\beta\gamma(1'_2)\gamma^{-1}\beta(1_2)) \cdot \nu(n) \\
&= 1_1 \cdot \mu(m) \otimes \gamma^{-1}\beta(1_2) \cdot \nu(n) \\
&= \mu(m) \otimes_{t_{\gamma^{-1}\beta}} \nu(n).
\end{aligned}$$

This shows that $(M \otimes_{t_{\gamma^{-1}\beta}} N, \mu \otimes \nu)$ is a left H -module and the right H -comodule condition is straightforward to check.

Next, we compute the compatibility condition as follows:

$$(h \cdot (m \otimes n))_{(0)} \otimes (h \cdot (m \otimes n))_{(1)}$$

$$\begin{aligned}
&= ((\gamma(h_1) \cdot m)_{(0)} \otimes (\gamma^{-1}\beta\gamma(h_2) \cdot n)_{(0)}) \otimes (\gamma^{-1}\beta\gamma(h_2) \cdot n)_{(1)}(\gamma(h_1) \cdot m)_{(1)} \\
&\stackrel{(3.1)}{=} (\gamma\xi(h_{121}) \cdot m_{(0)} \otimes \gamma^{-1}\beta\gamma\xi(h_{221}) \cdot n_{(0)}) \otimes ((\delta\gamma^{-1}\beta\gamma(h_{222})\xi^{-1}(n_{(1)})) \\
&\quad \gamma S^{-1}\gamma^{-1}\beta\gamma(h_{21}))((\beta\gamma(h_{122})\xi^{-1}(m_{(1)}))\alpha S^{-1}\gamma(h_{11})) \\
&= (\gamma(h_{12}) \cdot m_{(0)} \otimes \gamma^{-1}\beta\gamma\xi(h_{221}) \cdot n_{(0)}) \otimes (\delta\gamma^{-1}\beta\gamma\xi(h_{222})n_{(1)}) \\
&\quad ((\beta\gamma\xi^{-1}(\varepsilon(h_{21}1_1)1_2)\xi^{-1}(m_{(1)}))S^{-1}\alpha\gamma(h_{11})) \\
&= (\gamma(h_{12})\varepsilon(h_{21}1_1) \cdot m_{(0)} \otimes \gamma^{-1}\beta\gamma\xi(h_{221}) \cdot n_{(0)}) \otimes (\delta\gamma^{-1}\beta\gamma\xi(h_{222})n_{(1)}) \\
&\quad ((\beta\gamma\xi^{-1}(1_2)\xi^{-1}(m_{(1)}))S^{-1}\alpha\gamma(h_{11})) \\
&= (\gamma\xi^{-2}(h_{12}1_2) \cdot m_{(0)} \otimes \gamma^{-1}\beta\gamma(h_{21}) \cdot n_{(0)}) \otimes (\delta\gamma^{-1}\beta\gamma(h_{22})n_{(1)}) \\
&\quad ((\beta\gamma(1_3)\xi^{-1}(m_{(1)}))S^{-1}\alpha\gamma\xi^{-1}(h_{11}1_1)) \\
&= \gamma(h_{12}) \cdot (1_2 \cdot \mu^{-1}(m_{(0)})) \otimes \gamma^{-1}\beta\gamma(h_{21}) \cdot n_{(0)} \otimes (\delta\gamma^{-1}\beta\gamma(h_{22})n_{(1)}) \\
&\quad (((\beta(1_3)\xi^{-2}(m_{(1)}))\alpha S^{-1}(1_1))\alpha\gamma S^{-1}(h_{11})) \\
&= (\gamma\xi(h_{211}) \cdot m_{(0)} \otimes \gamma^{-1}\beta\gamma\xi(h_{212}) \cdot n_{(0)}) \otimes (\delta\gamma^{-1}\beta\gamma(h_{22})\xi^{-1}(n_{(1)}m_{(1)})) \\
&\quad S^{-1}\alpha\gamma(h_1) \\
&= \xi(h_{21}) \cdot (m \otimes n)_{(0)} \otimes \delta\gamma^{-1}\beta\gamma(h_{22})\xi^{-1}(m \otimes n)_{(1)}\alpha\gamma(S^{-1}(h_1)).
\end{aligned}$$

Thus $(M \otimes_{t_{\gamma^{-1}\beta}} N, \mu \otimes \nu) \in {}_H\mathcal{WMHYD}^H(\alpha\gamma, \delta\gamma^{-1}\beta\gamma)$. ■

Remark. Note that, if $(M, \mu) \in {}_H\mathcal{WMHYD}^H(\alpha, \beta)$, $(N, \nu) \in {}_H\mathcal{WMHYD}^H(\gamma, \delta)$ and $(P, \varsigma) \in {}_H\mathcal{WMHYD}^H(s, t)$, then the associativity constraint $a_{M,N,P}$ is

$$\begin{aligned}
a_{M,N,P} : (M \otimes_{t_{\gamma^{-1}\beta}} N) \otimes_{t_{s^{-1}\delta\gamma^{-1}\beta\gamma}} P &\rightarrow M \otimes_{t_{s^{-1}\gamma^{-1}\beta}} (N \otimes_{t_{s^{-1}\delta}} P) \\
(m \otimes n) \otimes p &\mapsto \mu(m) \otimes (n \otimes \varsigma^{-1}(p))
\end{aligned}$$

where $(M \otimes_{t_{\gamma^{-1}\beta}} N) \otimes_{t_{s^{-1}\delta\gamma^{-1}\beta\gamma}} P \in {}_H\mathcal{WMHYD}^H(\alpha\gamma s, ts^{-1}\delta\gamma^{-1}\beta\gamma s)$.

Denote $G = \text{Aut}_{wmHH}(H) \times \text{Aut}_{wmHH}(H)$ a group with multiplication as follows: for all $\alpha, \beta, \gamma, \delta \in \text{Aut}_{wmHH}(H)$,

$$(\alpha, \beta) * (\gamma, \delta) = (\alpha\gamma, \delta\gamma^{-1}\beta\gamma). \quad (4.1)$$

The unit of this group is (id, id) and $(\alpha, \beta)^{-1} = (\alpha^{-1}, \alpha\beta^{-1}\alpha^{-1})$.

The above proposition means that if $M \in {}_H\mathcal{WMHYD}^H(\alpha, \beta)$ and $N \in {}_H\mathcal{WMHYD}^H(\gamma, \delta)$, then $M \otimes N \in {}_H\mathcal{WMHYD}^H((\alpha, \beta) * (\gamma, \delta))$.

Proposition 4.2. The associativity constraints of monoidal category ${}_H\mathcal{WMHYD}^H$ are described as above. The left and right unit constraints $l_N : H_t \otimes_{t_{\gamma^{-1}}} N \rightarrow N$ and $r_M : M \otimes_{t_\beta} H_t \rightarrow M$ with $(N, \nu) \in {}_H\mathcal{WMHYD}^H(\gamma, \delta)$ and $(M, \mu) \in {}_H\mathcal{WMHYD}^H(\alpha, \beta)$ and their inverses are given by the formulas

$$\begin{aligned}
l_N(x \otimes_{t_{\gamma^{-1}}} n) &= \gamma^{-1}(x) \cdot n, \quad l_N^{-1}(n) = \varepsilon_t(1_1) \otimes_{t_{\gamma^{-1}}} \gamma^{-1}(1_2) \cdot \nu^{-2}(n); \\
r_M(m \otimes_{t_\beta} x) &= \widehat{\varepsilon}_s(\beta^{-1}(x)) \cdot m, \quad r_M^{-1}(m) = \beta^{-1}(1_1) \cdot \mu^{-2}(m) \otimes_{t_\beta} 1_2.
\end{aligned}$$

for all $x \in H_t$, $n \in N$ and $m \in M$.

Proof. Observe that $H_t \in {}_H\mathcal{WMHYD}^H(id, id)$, with left H -action $h \cdot x = \varepsilon_t(hx)$ and right H -coaction $\rho(x) = 1_2 \otimes S^{-1}(x1_1)$, for all $h \in H$, here ρ is the comodule structure map.

It is easy to get that H_t is a left H -module and a right H -comodule under \cdot and ρ . We just check $H_t \in {}_H\mathcal{WMHYD}^H(id, id)$.

On the one hand,

$$\begin{aligned}\rho(h \cdot x) &= \rho(\varepsilon_t(hx)) \\ &= 1_2 \otimes S^{-1}(1_1)S^{-1}((\xi^{-2}(h_1)x)S(\xi^{-1}(h_2))) \\ &= 1_2 \otimes S^{-1}(1_1)(\xi^{-1}(h_2)S^{-1}(\xi^{-2}(h_1)x)).\end{aligned}$$

On the other hand,

$$\begin{aligned}\rho(h \cdot x) &= \xi(h_{21}) \cdot 1_2 \otimes (h_{22}\xi^{-1}(S^{-1}(x1_1)))S^{-1}(h_1) \\ &= \varepsilon_t(\xi(h_{21})1_2) \otimes (h_{22}(S^{-1}(x1_1)))S^{-1}(h_1) \\ &= 1_2 \otimes \varepsilon(1_1h_{21})\xi(h_{22})S^{-1}(\xi^{-1}(h_1)x) \\ &= 1_2 \otimes (\widehat{\varepsilon}_t(1_1)\xi^{-1}(h_2))S^{-1}(\xi^{-1}(h_1)x) \\ &= 1_2 \otimes S^{-1}(1_1)(\xi^{-1}(h_2)S^{-1}(\xi^{-2}(h_1)x)).\end{aligned}$$

We just check the properties of l_N , the r_M cases left to reader. First we need to check l_N is both H -linear and H -colinear. Let $n \in N$. We have

$$\begin{aligned}l_N(h \cdot (x \otimes n)) &= l_M(\varepsilon_t(\alpha(h_1)x) \otimes h_2 \cdot n) \\ &= \alpha^{-1}(\varepsilon_t(h_1x)) \cdot (h_2 \cdot n) \\ &= (\xi^{-1}(h)\alpha^{-1}(x)) \cdot \mu(n) \\ &= h \cdot (\alpha^{-1}(x) \cdot n) \\ &= h \cdot l_M(x \otimes n).\end{aligned}$$

So l_N is H -linear, similar to the colinear case.

Next we check that

$$\begin{aligned}l_N(l_N^{-1}(n)) &= l_N(\varepsilon_t(1_1) \otimes \gamma^{-1}(1_2) \cdot \nu^{-2}(n)) \\ &= \gamma^{-1}(\varepsilon_t(1_1)1_2) \cdot \nu^{-1}(n) \\ &= 1 \cdot \nu^{-1}(n) = n.\end{aligned}$$

$$\begin{aligned}l_N^{-1}(l_N(x \otimes n)) &= l_N^{-1}(\gamma^{-1} \cdot n) \\ &= \varepsilon_t(1_1) \otimes \gamma^{-1}(1_2) \cdot \nu^{-2}(\gamma^{-1}(x) \cdot n) \\ &= \varepsilon_t(1_1x) \otimes \gamma^{-1}(1_2) \cdot \nu^{-1}(n) \\ &= (1'_1 1_1) \cdot x \otimes \gamma^{-1}(1'_2 1_2) \cdot n \\ &= x \otimes n.\end{aligned}$$

Finally we need to prove the following diagram commute.

$$\begin{array}{ccc}
(M \otimes_{t_\beta} H_t) \otimes_{t_{\gamma^{-1}\beta}} N & \xrightarrow{a_{M,H_t,N}} & M \otimes_{t_{\gamma^{-1}\beta}} (H_t \otimes_{t_{\gamma^{-1}}} N) \\
r_M \otimes id \downarrow & \swarrow id \otimes l_N & \\
M \otimes_{t_{\gamma^{-1}\beta}} N & &
\end{array}$$

$$\begin{aligned}
(r_M \otimes id)((m \otimes x) \otimes n) &= 1_1 \cdot (\beta^{-1}(\widehat{\varepsilon}_s(x)) \cdot m) \otimes \gamma^{-1}\beta(1_2) \cdot n \\
&= (1_1\beta^{-1}(\widehat{\varepsilon}_s(x)) \cdot \mu(m) \otimes \gamma^{-1}\beta(1_2) \cdot n \\
&= \widehat{\varepsilon}_s(\beta^{-1}(x))_1 \cdot \mu(m) \otimes \gamma^{-1}\beta(\varepsilon_t(\widehat{\varepsilon}_s(\beta^{-1}(x))_2)) \cdot n \\
&= 1_1 \cdot \mu(m) \otimes \gamma^{-1}\beta(1_2\beta^{-1}(x)) \cdot n \\
&= 1_1 \cdot \mu(m) \otimes \gamma^{-1}\beta(1_2) \cdot (\gamma^{-1}(x) \cdot \nu^{-1}(n)) \\
&= (id \otimes l_N) \circ a_{M,H_t,N}((m \otimes x) \otimes n).
\end{aligned}$$

This ends the proof. ■

Proposition 4.3. Let $(N, \nu) \in {}_H\mathcal{WMHYD}^H(\gamma, \delta)$ and $(\alpha, \beta) \in G$. Define $^{(\alpha, \beta)}N = N$ as vector space, with structures: for all $n \in N$ and $h \in H$.

$$h \triangleright n = \gamma^{-1}\beta\gamma\alpha^{-1}(h) \cdot n,$$

$$n \mapsto n_{<0>} \otimes n_{<1>} = n_{(0)} \otimes \alpha\beta^{-1}(n_{(1)}). \quad (4.2)$$

Then

$$^{(\alpha, \beta)}N \in {}_H\mathcal{WMHYD}^H((\alpha, \beta) * (\gamma, \delta) * (\alpha, \beta)^{-1}).$$

Proof. Obviously, the equations above define a module and a comodule action. In what follows, we show the compatibility condition:

$$\begin{aligned}
&(h \triangleright n)_{<0>} \otimes (h \triangleright n)_{<1>} \\
&= (\gamma^{-1}\beta\gamma\alpha^{-1}(h) \cdot n)_{(0)} \otimes \alpha\beta^{-1}((\gamma^{-1}\beta\gamma\alpha^{-1}(h) \cdot n)_{(1)}) \\
&= \gamma^{-1}\beta\gamma\alpha^{-1}\xi(h_{21}) \cdot n_{(0)} \otimes (\alpha\beta^{-1}\delta\gamma^{-1}\beta\gamma\alpha^{-1}(h_{22})\alpha\beta^{-1}\xi^{-1}(n_{(1)}))\alpha\gamma\alpha^{-1}S^{-1}(h_1) \\
&= \xi(h_{21}) \triangleright n_{<0>} \otimes (\alpha\beta^{-1}\delta\gamma^{-1}\beta\gamma\alpha^{-1}(h_{22})\xi^{-1}(n_{<1>}))\alpha\gamma\alpha^{-1}S^{-1}(h_1)
\end{aligned}$$

for all $n \in N$ and $h \in H$, that is $^{(\alpha, \beta)}N \in {}_H\mathcal{WMHYD}^H(\alpha\gamma\alpha^{-1}, \alpha\beta^{-1}\delta\gamma^{-1}\beta\gamma\alpha^{-1})$ ■

Remark. Let $(M, \mu) \in {}_H\mathcal{WMHYD}^H(\alpha, \beta)$, $(N, \nu) \in {}_H\mathcal{WMHYD}^H(\gamma, \delta)$, and $(s, t) \in G$. Then by the above proposition, we have:

$$^{(\alpha, \beta)* (s, t)}N = ^{(\alpha, \beta)}(^{(s, t)}N),$$

as objects in ${}_H\mathcal{WMHYD}^H(\alpha s \gamma s^{-1} \alpha^{-1}, \alpha \beta^{-1} s t^{-1} \delta \gamma^{-1} t s^{-1} \beta s \gamma s^{-1} \alpha^{-1})$ and

$$^{(s, t)}(M \otimes N) = ^{(s, t)}M \otimes ^{(s, t)}N,$$

as objects in ${}_H\mathcal{WMHYD}^H(s \alpha \gamma s^{-1}, s t^{-1} \delta \gamma^{-1} \beta \alpha^{-1} t \alpha \gamma s^{-1})$.

Proposition 4.4. Let $(M, \mu) \in {}_H\mathcal{WMHYD}^H(\alpha, \beta)$ and $(N, \nu) \in {}_H\mathcal{WMHYD}^H(\gamma, \delta)$, take ${}^M N = {}^{(\alpha, \beta)} N$ as explained in Subsection 1.2. Define a map $c_{M,N} : M \otimes N \rightarrow {}^M N \otimes M$ by

$$c_{M,N}(m \otimes n) = \nu(n_{(0)}) \otimes \beta^{-1}(n_{(1)}) \cdot \mu^{-1}(m). \quad (4.3)$$

for all $m \in M, n \in N$. Then $c_{M,N}$ is both an H -module map and an H -comodule map, and satisfies the following formulae (for $(P, \varsigma) \in {}_H\mathcal{WMHYD}^H(s, t)$):

$$a_{M \otimes N P, M, N}^{-1} \circ c_{M \otimes N, P} \circ a_{M, N, P}^{-1} = (c_{M, N P} \otimes id_N) \circ a_{M, N P, N}^{-1} \circ (id_M \otimes c_{N, P}), \quad (4.4)$$

$$a_{M N, M P, M} \circ c_{M, N \otimes P} \circ a_{M, N, P} = (id_M \otimes c_{M, P}) \circ a_{M N, M, P} \circ (c_{M, N} \otimes id_P). \quad (4.5)$$

Furthermore, if $(M, \mu) \in {}_H\mathcal{WMHYD}^H(\alpha, \beta)$ and $(N, \nu) \in {}_H\mathcal{WMHYD}^H(\gamma, \delta)$, then $c_{(s,t)M, (s,t)N} = c_{M,N}$, for all $(s, t) \in G$.

Proof. First, we prove that $c_{M,N}$ is an H -module map. Take $h \cdot (m \otimes n) = \gamma(h_1) \cdot m \otimes \gamma^{-1}\beta\gamma(h_2) \cdot n$ and $h \cdot (n \otimes m) = \gamma^{-1}\beta\gamma(h_1) \cdot n \otimes \beta^{-1}\delta\gamma^{-1}\beta\gamma(h_2) \cdot m$ as explained in Proposition 4.1.

$$\begin{aligned} & c_{M,N}(h \cdot (m \otimes n)) \\ &= \nu((\gamma^{-1}\beta\gamma(h_2) \cdot n)_{(0)}) \otimes \beta^{-1}((\gamma^{-1}\beta\gamma(h_2) \cdot n)_{(1)}) \cdot \mu^{-1}(\gamma(h_1) \cdot m) \\ &= \nu(\gamma^{-1}\beta\gamma\xi(h_{221}) \cdot n_{(0)}) \otimes \beta^{-1}(\delta\gamma^{-1}\beta\gamma\xi(h_{222})n_{(1)}) \\ & \quad \cdot ((\gamma S^{-1}\xi^{-1}(h_{21})\gamma\xi^{-2}(h_1)) \cdot \mu^{-1}(m)) \\ &= \nu(\gamma^{-1}\beta\gamma(\xi^{-2}(h_1)1_2) \cdot n_{(0)}) \otimes \beta^{-1}(\delta\gamma^{-1}\beta\gamma(\xi^{-2}(h_2)1_3)n_{(1)}) \\ & \quad \cdot (\gamma S^{-1}(1_1) \cdot \mu^{-1}(m)) \\ &= \nu(\gamma^{-1}\beta\gamma\xi^{-1}(h_1) \cdot (\gamma^{-1}\beta\gamma(1_2) \cdot \mu^{-1}(n_{(0)}))) \otimes (\beta^{-1}\delta\gamma^{-1}\beta\gamma\xi^{-1}(h_2) \\ & \quad (\beta^{-1}\delta\gamma^{-1}\beta\gamma(1_3)\beta^{-1}\xi^{-1}(n_{(1)})) \cdot (\gamma S^{-1}(1_1) \cdot \mu^{-1}(m))) \\ &= \nu(\gamma^{-1}\beta\gamma\xi^{-1}(h_1) \cdot (1_2 \cdot \nu^{-2}(n_{(0)}))) \otimes (\beta^{-1}\delta\gamma^{-1}\beta\gamma\xi^{-1}(h_2) \\ & \quad \beta^{-1}((\delta(1_3)\xi^{-2}(n_{(1)}))\gamma S^{-1}(1_1)) \cdot m) \\ &= \gamma^{-1}\beta\gamma(h_1) \cdot \nu(n_{(0)}) \otimes (\beta^{-1}\delta\gamma^{-1}\beta\gamma\xi^{-1}(h_2)\beta^{-1}(n_{(1)})) \cdot m \\ &= h \cdot c_{M \otimes N}(m \otimes n). \end{aligned}$$

Secondly, we check that $c_{M,N}$ is an H -comodule map as follows:

$$\begin{aligned} & \rho_{N \otimes M} \circ c_{M,N}(m \otimes n) \\ &= ((\nu(n_{(0)}))_{<0>} \otimes (\beta^{-1}(n_{(1)}) \cdot \mu^{-1}(m))_{(0)}) \otimes (\beta^{-1}(n_{(1)}) \cdot \mu^{-1}(m))_{(1)} \\ & \quad (\nu(n_{(0)}))_{<1>} \\ &= (n_{(0)} \otimes \beta^{-1}\xi(n_{(1)21}) \cdot \mu^{-1}(m_{(0)})) \otimes (\xi(n_{(1)22})\xi^{-1}(m_{(1)})) \\ & \quad (\alpha\beta^{-1}S^{-1}\xi(n_{(1)12})\alpha\beta^{-1}\xi(n_{(1)11})) \\ &= \varepsilon(n_{(1)11}1'_1 1_2)(n_{(0)} \otimes \beta^{-1}(n_{(1)12}1'_2) \cdot \mu^{-1}(m_{(0)})) \\ & \quad \otimes (n_{(1)2}\xi^{-1}(m_{(1)}))\alpha\beta^{-1}S^{-1}(1_1) \\ &= (n_{(0)} \otimes \beta^{-1}\xi^{-1}(n_{(1)1}1_2) \cdot \mu^{-1}(m_{(0)})) \\ & \quad \otimes (n_{(1)2}\xi^{-1}(m_{(1)}))\alpha\beta^{-1}S^{-1}(1_1) \end{aligned}$$

$$\begin{aligned}
&= (n_{(0)} \otimes \beta^{-1}(n_{(1)1}) \cdot \mu^{-1}(m_{(0)}) \otimes \xi(n_{(1)2})m_{(1)}) \\
&= (\nu(n_{(0)(0)}) \otimes \beta^{-1}(n_{(0)(1)}) \cdot \mu^{-1}(m_{(0)})) \otimes n_{(1)}m_{(1)} \\
&= (c_{M,N} \otimes id)((m_{(0)} \otimes n_{(0)}) \otimes n_{(1)}m_{(1)}) = (c_{M,N} \otimes id)\rho(m \otimes n).
\end{aligned}$$

Finally we will check Eqs.(4.4) and (4.5). On the one hand,

$$\begin{aligned}
&a_{M \otimes N, P, M, N}^{-1} \circ c_{M \otimes N, P} \circ a_{M, N, P}^{-1}(m \otimes (n \otimes p)) \\
&= a_{M \otimes N, P, M, N}^{-1} \circ c_{M \otimes N, P}((\mu^{-1}(m) \otimes n) \otimes \varsigma(p)) \\
&= a_{M \otimes N, P, M, N}^{-1}(\varsigma^2(p_{(0)}) \otimes \gamma^{-1}\beta^{-1}\gamma\delta^{-1}\xi(p_{(1)}) \cdot (\mu^{-2}(m) \otimes \nu^{-1}(n))) \\
&= (c_{M, N, P} \otimes id_N)((\mu^{-1}(m) \otimes \varsigma(p_{(0)})) \otimes \delta^{-1}\xi(p_{(1)}) \cdot n) \\
&= (c_{M, N, P} \otimes id_N) \circ a_{M, N, P, N}^{-1} \circ (id_M \otimes c_{N, P})(m \otimes (n \otimes p)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&a_{M, N, M, P, M} \circ c_{M, N \otimes P} \circ a_{M, N, P}((m \otimes n) \otimes p) \\
&= a_{M, N, M, P, M} \circ c_{M, N \otimes P}(\mu(m) \otimes (n \otimes \varsigma^{-1}(p))) \\
&= \nu^2(n_{(0)}) \otimes \varsigma(\varsigma^{-1}(p)_{(0)}) \otimes (\beta^{-1}(\varsigma^{-1}(p)_{(1)}) \cdot (\beta^{-1}\xi^{-1}(n_{(1)}) \cdot \xi^{-1}(m))) \\
&= (id_{M, N} \otimes c_{M, P})(\nu^2(n_{(0)}) \otimes \beta^{-1}(n_{(1)}) \cdot \mu^{-1}(m)) \otimes p \\
&= (id_{M, N} \otimes c_{M, P}) \circ a_{M, N, M, P} \circ (c_{M, N} \otimes id_P)((m \otimes n) \otimes p).
\end{aligned}$$

The proof is completed. ■

Lemma 4.5. The map $c_{M,N}$ defined by $c_{M,N}(m \otimes n) = \nu(n_{(0)}) \otimes \beta^{-1}(n_{(1)}) \cdot \mu^{-1}(m)$ is bijective; with inverse

$$c_{M,N}^{-1}(n \otimes m) = \beta^{-1}(S(n_{(1)})) \cdot \mu^{-1}(m) \otimes \nu(n_{(0)}).$$

Proof. First, we prove $c_{M,N}c_{M,N}^{-1} = id$. For all $m \in M, n \in N$, we have

$$\begin{aligned}
&c_{M,N}c_{M,N}^{-1}(n \otimes m) \\
&= c_{M,N}(\beta^{-1}S(n_{(1)}) \cdot \mu^{-1}(m) \otimes \nu(n_{(0)})) \\
&= \nu(\nu(n_{(0)(0)}) \otimes \beta^{-1}(\nu(n_{(0)(1)}) \cdot \mu^{-1}(\beta^{-1}S(n_{(1)}) \cdot \mu^{-1}(m))) \\
&= \nu^2(n_{(0)(0)}) \otimes \beta^{-1}((n_{(0)(1)})S\xi^{-1}(n_{(1)})) \cdot \mu^{-1}(m) \\
&= \alpha\gamma^{-1}\beta^{-1}\gamma\delta^{-1}\beta\alpha^{-1}(1_1) \triangleright \nu(n_{<0>}) \otimes \alpha^{-1}(1_2\varepsilon_t(n_{<1>})) \cdot m \\
&= 1_1 \triangleright \nu(n) \otimes \beta^{-1}\delta^{-1}\gamma^{-1}\beta\gamma\alpha^{-1}(1_2) \cdot m \\
&= n \otimes m.
\end{aligned}$$

The fact that $c_{M,N}^{-1}c_{M,N} = id$ is similar. This completes the proof. ■

Let H be a weak monoidal Hom-Hopf algebra and $G = Aut_{wmHH}(H) \times Aut_{wmHH}(H)$. Define $\mathcal{WMHYD}(H)$ as the disjoint union of all ${}_H\mathcal{WMHYD}^H(\alpha, \beta)$ with $(\alpha, \beta) \in G$. If we endow $\mathcal{WMHYD}(H)$ with tensor product shown in Proposition 4.1, then $\mathcal{WMHYD}(H)$ becomes a monoidal category with unit H_t .

Define a group homomorphism $\varphi : G \rightarrow \text{Aut}(\mathcal{WMHYD}(H))$, $(\alpha, \beta) \mapsto \varphi(\alpha, \beta)$ on components as follows:

$$\begin{aligned}\varphi_{(\alpha, \beta)} : {}_H\mathcal{WMHYD}^H(\gamma, \delta) &\rightarrow {}_H\mathcal{WMHYD}^H((\alpha, \beta) * (\gamma, \delta) * (\alpha, \beta)^{-1}), \\ \varphi_{(\alpha, \beta)}(N) &= {}^{(\alpha, \beta)}N,\end{aligned}$$

and the functor $\varphi_{(\alpha, \beta)}$ acts as identity on morphisms.

The braiding in $\mathcal{WMHYD}(H)$ is given by the family $\{c_{M, N}\}$ in Proposition 4.4. So we get the following main theorem of this article.

Theorem 4.6. $\mathcal{WMHYD}(H)$ is a braided T -category over G .

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